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A note on the global existence of small amplitude solutions to a generalized Davey–Stewartson system

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Abstract
In this paper, we are interested in the Cauchy problem for a generalized Davey–Stewartson (GDS) system. We establish the global time existence of small mass solutions for the GDS system in the elliptic–hyperbolic–hyperbolic case.

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1. Introduction

We continue with our study of the elliptic–hyperbolic–hyperbolic (EHH) case of the generalized Davey–Stewartson (GDS) system that was initiated in [1]. The GDS system was introduced in [2] as a model for wave propagation in an infinite elastic medium with coupled stresses. The resulting system of partial differential equations was put in a non-dimensional form in [1] as

$$\begin{aligned} &\mathrm{i}u_t + \Delta u = \chi |u|^2 u + b(\alpha\phi_{1,x} + \phi_{2,y})u, \\ &\phi_{1,xx} - \phi_{1,yy} - \frac{c_1^2 - c_2^2}{c_2\sqrt{c_g^2 - c_2^2}}\phi_{2,xy} = \alpha(|u|^2)_x, \\ &\phi_{2,xx} - \frac{c_1^2\left(c_g^2 - c_1^2\right)}{c_2^2\left(c_g^2 - c_2^2\right)}\phi_{2,yy} - \frac{c_1^2 - c_2^2}{c_2\sqrt{c_g^2 - c_2^2}}\phi_{1,xy} = (|u|^2)_y, \\ &u(0, x, y) = u_0(x, y), \end{aligned} \tag{1}$$

where $u : [0, \infty) \times \mathbb{R}^2 \to \mathbb{C}$, $\phi_i : [0, \infty) \times \mathbb{R}^2 \to \mathbb{R} (i = 1, 2)$ and the constants appearing above are all real. It was observed in [3] that the parameters have the order $c_g > c_1 > c_2 > 0$ for the specific materials Brass LS-62 and Bronze Brof 35 [4]. This corresponds to the EHH case of the GDS system according to the classification given in [3]. In order to obtain a well-posed problem, this system was complemented with radiation-type boundary conditions for the variables ϕ_1 and ϕ_2 and null condition at infinity for the variable u. The radiation-type boundary conditions are given along the characteristics ξ_1 and ξ_2 as

$$\lim_{\xi_1\to\infty}\phi_1(x,y)=\lim_{\xi_2\to\infty}\phi_1(x,y)=0,\qquad \lim_{\xi_1\to\infty}\phi_2(x,y)=\lim_{\xi_2\to\infty}\phi_2(x,y)=0.$$

Here, the characteristics ξ_1 and ξ_2 are defined as

$$\xi_1=-r_1x+y,\qquad \xi_2=r_1x+y(r_1=c_1/c_2). \tag{2}$$

In fact, the system of $(1)_2$ and $(1)_3$ has many sets of characteristics among which we single out $\eta_1=-r_2x+y$ and $\eta_2=r_2x+y\big(r_2=\big[\big(c_g^2-c_1^2\big)/\big(c_g^2-c_2^2\big)\big]^{1/2}\big)$ [1].

The aim of our paper is to establish the existence of the global solutions to the Cauchy problem for (1) when the initial data u_0 are in the Sobolev space $H^1(\mathbb{R}^2)$ and have a small $L^2(\mathbb{R}^2)$ norm. The first step in this direction was taken in [1] where system (1) was reduced to a nonlinear Schrödinger (NLS) equation with non-local terms for $r_1\neq r_2$:

$$\begin{aligned}&\mathrm{i}u_t+\Delta u=\chi|u|^2u+bu\{\alpha[\mathcal{K}_1(\alpha(|u|^2)_x,(|u|^2)_y)]_x+[\mathcal{K}_2(\alpha(|u|^2)_x,(|u|^2)_y)]_y\},\\&u(0,x,y)=u_0(x,y).\end{aligned} \tag{3}$$

The non-local operators $\mathcal{K}_1$ and $\mathcal{K}_2$ that appear above are convolution-type operators that depend on the characteristics ξ_1 and ξ_2 given in (2). Their explicit expressions are recalled in the appendix. The main technical difficulty in the analysis of the Cauchy problem under consideration originates from the lack of regularity of these operators. In the literature, this difficulty has been circumvented for the Davey–Stewartson (DS) equations either by assuming a smooth class of initial data or by assuming that the data are small with respect to some norm [5–16]. Here, in order to establish the global existence of weak solutions, we will follow the approach given in [5] and regularize both the nonlinearities and the initial data. This type of regularization also allows us to conclude that the Hamiltonian is a decreasing function of time. In contrast, for the DS system, Tsutsumi [7] used a different type of regularization and obtained a pseudo-conformal inequality and the $L^p(\mathbb{R}^2)$ decay of the weak solutions.

Our paper consists of three parts. In the second section, we obtain explicit estimates of the solutions of the hyperbolic system, the second and third equations of (1), using the representation obtained in [1]. In the third section, we introduce the regularized equation and prove the global existence and uniqueness of its solutions by a fixed point argument. In the final section, we prove our main theorem, theorem 1, by passing to the limit.

2. Estimates on the solutions of the hyperbolic system

We start by considering two asymmetrically coupled linear wave equations $(1)_2$ and $(1)_3$:

$$\phi_{1,xx}-\phi_{1,yy}-\beta\phi_{2,xy}=f,\qquad \phi_{2,xx}-\lambda\phi_{2,yy}-\beta\phi_{1,xy}=g, \tag{4}$$

where $f=\alpha(|u|^2)_x$ and $g=(|u|^2)_y$. The coefficients that appear in (4) are given by $\lambda=a_1^2c_1^2\big/\big(a_2^2c_2^2\big)$ and $\beta=c_1^2-c_2^2/(a_2c_2)$, where $a_1=\big(c_g^2-c_1^2\big)^{1/2}$ and $a_2=\big(c_g^2-c_2^2\big)^{1/2}$.

An integral representation for the solutions ϕ_1 and ϕ_2 of equations (4) was derived in [1]. This representation is recalled in (A.2). Since the representation is valid for $r_1\neq r_2$, we assume that $c_1\neq c_2$. When $u\in H^1(\mathbb{R}^2)$, the functions f and g that appear in (4) do not necessarily fall into $L^2(\mathbb{R}^2)$ but to $L^1(\mathbb{R}^2)$. Our main aim in this section is to derive some estimates on ϕ_1 and ϕ_2, in propositions 1 and 2, that depend only on $L^1(\mathbb{R}^2)$-norms of f and g.

To start with, we derive a point-wise bound for ϕ_1 and ϕ_2 using the representation of the solutions given in (A.1) and the fact that the Heaviside function is bounded by 1, namely:

Proposition 1. *Let ϕ_1 and ϕ_2 be solutions of the coupled linear wave system (4); then*

$$\sup_{(x,y)\in\mathbb{R}^2} |\phi_1(x,y)| \leqslant \left(\frac{c_1^2}{2c_g^2 r_1} + \frac{a_1^2}{c_g^2 r_2}\right)\|f\|_{L^1(\mathbb{R}^2)} + \frac{3c_2 a_2}{2c_g^2}\|g\|_{L^1(\mathbb{R}^2)},$$

$$\sup_{(x,y)\in\mathbb{R}^2} |\phi_2(x,y)| \leqslant \frac{3c_2 a_2}{2c_g^2}\|f\|_{L^1(\mathbb{R}^2)} + \left(\frac{a_2^2}{2c_g^2 r_1} + \frac{c_2^2}{c_g^2 r_2}\right)\|g\|_{L^1(\mathbb{R}^2)}.$$

Further estimates on ϕ_1 and ϕ_2 involving their partial derivatives are more intricate in nature. In particular, the energy of the original system (1), which is given by

$$E = \int_{\mathbb{R}^2}\left\{|\nabla u|^2 + \frac{\chi}{2}|u|^4 + \frac{b}{2}\mathcal{H}_\phi\right\}\mathrm{d}x\,\mathrm{d}y,$$

has $\mathcal{H}_\phi = (\phi_{1,x})^2 - (\phi_{1,y})^2 + (\phi_{2,x})^2 - \lambda(\phi_{2,y})^2 - \beta(\phi_{1,y}\phi_{2,x} + \phi_{1,x}\phi_{2,y})$. In order to control the energy, one needs to control $\mathcal{H}_\phi$. This term when written in terms of the characteristic coordinates ξ_1 and ξ_2 given in (2) takes the form

$$\begin{aligned}\mathcal{H}_\phi(\xi_1,\xi_2) = {}& \left(r_1^2-1\right)\left[\left(\phi_{1,\xi_1}\right)^2 + \left(\phi_{1,\xi_2}\right)^2\right] + \left(r_1^2-\lambda\right)\left[\left(\phi_{2,\xi_1}\right)^2 + \left(\phi_{2,\xi_2}\right)^2\right] \\ & - 2\left(r_1^2+1\right)\phi_{1,\xi_1}\phi_{1,\xi_2} - 2\left(r_1^2+\lambda\right)\phi_{2,\xi_1}\phi_{2,\xi_2} + 2\beta r_1\left(\phi_{1,\xi_1}\phi_{2,\xi_1} - \phi_{1,\xi_2}\phi_{2,\xi_2}\right). \end{aligned} \quad (5)$$

Integrals of this quantity involve integrals of ϕ_{i,ξ_j}'s $(i,j = 1,2)$, which in turn involve quadruple integration of the functions f and g on various characteristic directions. These computations are greatly simplified due to the following observation:

$$\int_{\mathbb{R}^4} |v_1(\xi_1,\xi_2')||v_2(\xi_1',\xi_2)|\mathrm{d}\xi_1'\,\mathrm{d}\xi_2'\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2 \leqslant 4r_1^2\left(\int_{\mathbb{R}^2}|v_1(x,y)|\mathrm{d}x\,\mathrm{d}y\right)\left(\int_{\mathbb{R}^2}|v_2(x,y)|\mathrm{d}x\,\mathrm{d}y\right). \quad (6)$$

Proposition 2. *Let ϕ_1 and ϕ_2 be solutions of (4); then*

$$H_\phi = \int_{\mathbb{R}^2}|\mathcal{H}_\phi|\mathrm{d}x\,\mathrm{d}y \leqslant b_1\|f\|^2_{L^1(\mathbb{R}^2)} + \frac{b_1}{r_1 r_2}\|g\|^2_{L^1(\mathbb{R}^2)} + b_2\|f\|_{L^1(\mathbb{R}^2)}\|g\|_{L^1(\mathbb{R}^2)}, \quad (7)$$

where

$$b_1 = (a_1a_2 + c_1c_2)\frac{\left|a_2^2 - c_1^2\right|}{2c_g^4} + \frac{2a_2c_1^2}{c_g^2}(|d_1| + d_2),$$

$$b_2 = \frac{2a_2c_2}{c_g^4}\left|a_2^2 - c_1^2\right| + \frac{2c_1}{c_g^2 r_2}(a_1a_2 + c_1c_2)(|d_1| + d_2),$$

$$d_1 = \frac{a_1a_2\left(c_1^2+c_2^2\right) - c_1c_2\left(a_1^2+a_2^2\right)}{4c_g^2c_1^2a_2}, \qquad d_2 = \frac{a_1a_2\left(c_1^2+c_2^2\right) + c_1c_2\left(a_1^2+a_2^2\right)}{4c_g^2c_1^2a_2}.$$

Proof. From (A.2), we deduce that

$$\phi_{1,\xi_j}(\xi_1,\xi_2) = -\frac{c_2}{2a_2}I_{1,\xi_j} + \frac{a_1}{2c_1}\left(I_{2,\xi_j} + I_{3,\xi_j}\right), \phi_{2,\xi_j}(\xi_1,\xi_2) = \frac{1}{2r_1}\left(I_{1,\xi_j} - I_{2,\xi_j} + I_{3,\xi_j}\right), \quad (8)$$

where

$$I_{1,\xi_1} = \frac{a_2}{2c_g^2 r_1}(a_2g_2 - c_1f_2), \qquad I_{2,\xi_1} = -A_1,$$

$$I_{3,\xi_1} = -A_2, \qquad I_{1,\xi_2} = \frac{a_2}{2c_g^2 r_1}(a_2g_1 - c_1f_1), \quad (9)$$

$$I_{2,\xi_2} = \frac{c_2}{2c_g^2 r_2}(c_2g_1 - a_1f_1) - r_8A_1, \qquad I_{3,\xi_2} = \frac{c_2}{2c_g^2 r_2}(a_1f_1 + c_2g_1) - r_7A_2.$$

The functions f_1, f_2, g_1, g_2, A_1 and A_2 in (9) are defined by

$$f_1 = \int_{\xi_1}^{\infty} f(\xi_1', \xi_2)\, d\xi_1', \qquad f_2 = \int_{\xi_2}^{\infty} f(\xi_1, \xi_2')\, d\xi_2',$$

$$g_1 = \int_{\xi_1}^{\infty} g(\xi_1', \xi_2)\, d\xi_1', \qquad g_2 = \int_{\xi_2}^{\infty} g(\xi_1, \xi_2')\, d\xi_2',$$

$$A_1 = \int_{\mathbb{R}^2} \delta(\xi_1' - \xi_1 + r_8(\xi_2' - \xi_2)) H(-r_6(\xi_2' - \xi_2)) h_1(\xi_1', \xi_2')\, d\xi_1'\, d\xi_2',$$

$$A_2 = \int_{\mathbb{R}^2} \delta(\xi_1' - \xi_1 + r_7(\xi_2' - \xi_2)) H(r_5(\xi_2' - \xi_2)) h_2(\xi_1', \xi_2')\, d\xi_1'\, d\xi_2',$$

where $\delta(.)$ is the Dirac delta distribution, H is the Heaviside function and the notations f_i and g_i denote integrals of the functions f and g with respect to the variable ξ_j. h_1 and h_2 are defined in (A.3). The right-hand sides of A_1 and A_2 can be written in two equivalent forms as

$$\begin{aligned} A_1 &= \int_{\mathbb{R}} H(-r_6(\xi_2' - \xi_2)) h_1(\xi_1 - r_8(\xi_2' - \xi_2), \xi_2')\, d\xi_2' \\ &= r_7 \int_{\mathbb{R}} H(r_5(\xi_1' - \xi_1)) h_1(\xi_1', \xi_2 - r_7(\xi_1' - \xi_1))\, d\xi_1' \end{aligned} \tag{10}$$

and

$$\begin{aligned} A_2 &= \int_{\mathbb{R}} H(r_5(\xi_2' - \xi_2)) h_2(\xi_1 - r_7(\xi_2' - \xi_2), \xi_2')\, d\xi_2' \\ &= r_8 \int_{\mathbb{R}} H(-r_6(\xi_1' - \xi_1)) h_2(\xi_1', \xi_2 - r_8(\xi_1' - \xi_1))\, d\xi_1', \end{aligned} \tag{11}$$

respectively.

Substituting (8) into equation (5), we find

$$\begin{aligned} \mathcal{H}_\phi(\xi_1, \xi_2) = {} & \frac{a_2^2 - c_1^2}{4c_g^4 r_1^2}\left(c_1^2 f_1 f_2 - a_2^2 g_1 g_2\right) + \frac{a_2 c_2\left(c_1^2 - a_2^2\right)}{4c_g^4 r_1}(f_1 g_2 - f_2 g_1) \\ & + \frac{4a_1^2\left(c_1^2 - a_2^2\right)}{\left(c_1^2 - c_2^2\right)c_g^2} A_1 A_2 + [d_1(c_1 f_1 + a_2 g_1) + d_2 r_8(c_1 f_2 - a_2 g_2)]A_1 \\ & + [d_2(c_1 f_1 + a_2 g_1) + d_1 r_7(c_1 f_2 - a_2 g_2)]A_2. \end{aligned}$$

We choose suitable forms of A_1 and A_2 from (10) and (11) to use (6) in H_ϕ; then we obtain

$$\begin{aligned} 2r_1 H_\phi \leqslant {} & \frac{\left|a_2^2 - c_1^2\right|}{4c_g^4 r_1^2}\left[c_1^2 J_1(f) + a_2^2 J_2(g) + a_2 c_1 (J_1(g) + J_2(f))\right] + |d_1|[c_1(J_3(f) + J_4(f)) \\ & + a_2(J_3(g) + J_4(g))] + d_2[c_1(J_5(f) + J_6(f)) + a_2(J_5(g) + J_6(g))] \\ & + r_6^2\left|a_2^2 - c_1^2\right| J_7 / c_2^2, \end{aligned}$$

where the J_k's ($k = 1, \ldots, 7$) are given in (A.4). To find an upper bound for H_ϕ, we apply (6) to each integral J_k above. J_1 and J_2 can be estimated directly by (6). For the rest of the J_k terms, we will apply coordinate transformations that will simplify the four-fold integrals to products of double integrals. In the following arguments, v stands for either f or g.

(i) If we apply the translational coordinate transformation in the integrals J_3

$$\xi_1 = \bar{\bar{\xi}}_1 + r_8(\bar{\bar{\xi}}_2 - \bar{\xi}_2), \qquad \xi_2 = \bar{\xi}_2, \qquad \xi_1' = \bar{\xi}_1, \qquad \xi_2' = \bar{\bar{\xi}}_2,$$

we obtain $J_3(v) \leqslant \int_{\mathbb{R}^4} |v(\bar{\xi}_1, \bar{\xi}_2)||h_1(\bar{\bar{\xi}}_1, \bar{\bar{\xi}}_2)| d\bar{\xi}_1\, d\bar{\xi}_2\, d\bar{\bar{\xi}}_1\, d\bar{\bar{\xi}}_2$.

(ii) Similarly, the transformation

$$\xi_1 = \bar{\xi}_1, \qquad \xi_2 = \bar{\bar{\xi}}_2 + r_8(\bar{\bar{\xi}}_1 - \bar{\xi}_1), \qquad \xi_1' = \bar{\bar{\xi}}_1, \qquad \xi_2' = \bar{\xi}_2$$

is used in J_4 and leads to $J_4(v) \leqslant \int_{\mathbb{R}^4} |v(\bar{\xi}_1, \bar{\xi}_2)||h_2(\bar{\bar{\xi}}_1, \bar{\bar{\xi}}_2)| \mathrm{d}\bar{\xi}_1 \, \mathrm{d}\bar{\xi}_2 \, \mathrm{d}\bar{\bar{\xi}}_1 \, \mathrm{d}\bar{\bar{\xi}}_2$.

(iii) If we apply the following transformation in J_5:

$$\xi_1 = \bar{\xi}_1, \qquad \xi_2 = \bar{\bar{\xi}}_2 + r_7(\bar{\bar{\xi}}_1 - \bar{\xi}_1), \qquad \xi_1' = \bar{\bar{\xi}}_1, \qquad \xi_2' = \bar{\xi}_2,$$

we obtain $J_5(v) \leqslant \int_{\mathbb{R}^4} |v(\bar{\xi}_1, \bar{\xi}_2)||h_1(\bar{\bar{\xi}}_1, \bar{\bar{\xi}}_2)| \mathrm{d}\bar{\xi}_1 \, \mathrm{d}\bar{\xi}_2 \, \mathrm{d}\bar{\bar{\xi}}_1 \, \mathrm{d}\bar{\bar{\xi}}_2$.

(iv) The translational coordinate transformation in J_6

$$\xi_1 = \bar{\bar{\xi}}_1 + r_7(\bar{\bar{\xi}}_2 - \bar{\xi}_2), \qquad \xi_2 = \bar{\xi}_2, \qquad \xi_1' = \bar{\xi}_1, \qquad \xi_2' = \bar{\bar{\xi}}_2$$

gives $J_6(v) \leqslant \int_{\mathbb{R}^4} |v(\bar{\xi}_1, \bar{\xi}_2)||h_2(\bar{\bar{\xi}}_1, \bar{\bar{\xi}}_2)| \mathrm{d}\bar{\xi}_1 \, \mathrm{d}\bar{\xi}_2 \, \mathrm{d}\bar{\bar{\xi}}_1 \, \mathrm{d}\bar{\bar{\xi}}_2$.

(v) Finally, the transformation that is used in J_7 is given by

$$\xi_1 = -\frac{\bar{\xi}_1}{r_4 r_6} + \frac{\bar{\bar{\xi}}_1}{r_3 r_5} - \frac{\bar{\xi}_2 - \bar{\bar{\xi}}_2}{r_3 r_6}, \qquad \xi_1' = \bar{\xi}_1,$$

$$\xi_2 = \frac{\bar{\xi}_1 - \bar{\bar{\xi}}_1}{r_3 r_6} + \frac{\bar{\xi}_2}{r_3 r_5} - \frac{\bar{\bar{\xi}}_2}{r_4 r_6}, \qquad \xi_2' = \bar{\xi}_2,$$

and leads to $J_7 \leqslant \frac{1}{r_4 r_6} \int_{\mathbb{R}^4} |h_1(\bar{\xi}_1, \bar{\xi}_2))||h_2(\bar{\bar{\xi}}_1, \bar{\bar{\xi}}_2)| \mathrm{d}\bar{\xi}_1 \, \mathrm{d}\bar{\xi}_2 \, \mathrm{d}\bar{\bar{\xi}}_1 \, \mathrm{d}\bar{\bar{\xi}}_2$.

Combining the above estimates, we arrive at

$$\begin{aligned}\frac{H_\phi}{2r_1} \leqslant \frac{|a_2^2 - c_1^2|}{4c_g^4 r_1^2}(c_1\|f\|_{L^1(\mathbb{R}^2)} + a_2\|g\|_{L^1(\mathbb{R}^2)})^2 + \frac{r_2|a_2^2 - c_1^2|}{c_1 c_2}\|h_1\|_{L^1(\mathbb{R}^2)}\|h_2\|_{L^1(\mathbb{R}^2)} \\ + (|d_1| + d_2)(c_1\|f\|_{L^1(\mathbb{R}^2)} + a_2\|g\|_{L^1(\mathbb{R}^2)})(\|h_1\|_{L^1(\mathbb{R}^2)} + \|h_2\|_{L^1(\mathbb{R}^2)}).\end{aligned} \tag{12}$$

Since $\|h_i\|_{L^1(\mathbb{R}^2)} \leqslant c_2\big(a_1\|f\|_{L^1(\mathbb{R}^2)} + c_2\|g\|_{L^1(\mathbb{R}^2)}\big)\big/(2c_g^2 r_2)$, (12) becomes (7). □

3. The regularized equation

Now, we consider the Cauchy problem for (3) with an initial value u_0 being in the appropriate Sobolev space $H^s(\mathbb{R}^2)$. The natural choice of $s = 1$ does not result in enough regularity for the functions f and g that were considered in the previous section, since we need to take their traces on various characteristic lines. In order to seek solutions in a smoother function space, we regularize the equation as is done for the DS system in [5]. By a fixed point argument in $H^2(\mathbb{R}^2)$, we will first establish the global existence and uniqueness of solutions for the regularized problem: proposition 4. Next, we will establish bounds on solutions independent of the regularization parameter $\varepsilon > 0$ when the initial data of the original problem have small enough mass: proposition 5.

For $\varepsilon > 0$, the regularized system is given by

$$\begin{aligned} &\mathrm{i}u_t^\varepsilon + \mathrm{i}\varepsilon\Delta^2 u_t^\varepsilon + \Delta u^\varepsilon = \chi|u^\varepsilon|^2 u^\varepsilon + bu^\varepsilon\big[\alpha\big(\phi_1^\varepsilon\big)_x + \big(\phi_2^\varepsilon\big)_y\big], \\ &u^\varepsilon(0, x, y) = u_0^\varepsilon(x, y), \qquad u_0^\varepsilon \in H^2(\mathbb{R}^2), \end{aligned} \tag{13}$$

where $\phi_1^\varepsilon = \mathcal{K}_1\left(\alpha(|u^\varepsilon|^2)_x, (|u^\varepsilon|^2)_y\right)$ and $\phi_2^\varepsilon = \mathcal{K}_2\left(\alpha(|u^\varepsilon|^2)_x, (|u^\varepsilon|^2)_y\right)$. The key ingredient in obtaining the global existence of solutions is the conservation of mass and energy. Here, the mass is given by

$$M_\varepsilon(t) \equiv \int_{\mathbb{R}^2} (|u^\varepsilon(t)|^2 + \varepsilon|\Delta u^\varepsilon(t)|^2) \, \mathrm{d}x \, \mathrm{d}y = M_\varepsilon(0), \qquad \forall t \in \mathbb{R}_+, \tag{14}$$

and the energy is given by

$$E_\varepsilon(t) \equiv \int_{\mathbb{R}^2} \left(|\nabla u^\varepsilon(t)|^2 + \frac{\chi}{2}|u^\varepsilon(t)|^4 + \frac{b}{2}\mathcal{H}_\phi^\varepsilon(t) \right) \mathrm{d}x\, \mathrm{d}y = E_\varepsilon(0), \qquad \forall t \in \mathbb{R}_+, \tag{15}$$

with $\mathcal{H}_\phi^\varepsilon = \left(\phi_{1,x}^\varepsilon\right)^2 - \left(\phi_{1,y}^\varepsilon\right)^2 + \left(\phi_{2,x}^\varepsilon\right)^2 - \lambda\left(\phi_{2,y}^\varepsilon\right)^2 - \beta\left(\phi_{1,y}^\varepsilon\phi_{2,x}^\varepsilon + \phi_{1,x}^\varepsilon\phi_{2,y}^\varepsilon\right)$.

In the same spirit as propositions 1 and 2, we have the following *a priori* estimates for the regularized equation (13).

Proposition 3. *The following inequalities are satisfied:*

(i) $\|\phi_1^\varepsilon\|_{L^\infty(\mathbb{R}^2)} \leqslant C_1\|u^\varepsilon\|_{H^1(\mathbb{R}^2)}^2$, $\|\phi_2^\varepsilon\|_{L^\infty(\mathbb{R}^2)} \leqslant C_2\|u^\varepsilon\|_{H^1(\mathbb{R}^2)}^2$, *where* $C_1 = [|\alpha|(2a_1a_2 + c_1c_2) + 3a_2c_2]/c_g^2$ *and* $C_2 = [3a_1c_1|\alpha| + a_1a_2 + 2c_1c_2]/\left(c_g^2 r_1 r_2\right)$,

(ii) $\|\mathcal{H}_\phi^\varepsilon\|_{L^1\mathbb{R}^2)} \leqslant 4\left(b_1\alpha^2 + \frac{b_1}{r_1r_2} + b_2\alpha\right)\|\nabla u^\varepsilon\|_{L^2(\mathbb{R}^2)}^2\|u^\varepsilon\|_{L^2(\mathbb{R}^2)}^2$.

Now we are ready to state and prove the global existence of a unique solution of the regularized problem (13).

Proposition 4. *For every* $u_0^\varepsilon \in H^2(\mathbb{R}^2)$*, the Cauchy problem (13) has a unique solution* u^ε *in* $\mathcal{C}(\mathbb{R}_+; H^2(\mathbb{R}^2))$.

□

Proof. We assume that $v(t) = \Sigma_\varepsilon(t)v_0$ solves the regularized linear equation:

$$\mathrm{i}v_t + \mathrm{i}\varepsilon\Delta^2 v_t + \Delta v = 0, \qquad v(0) = v_0.$$

We define the map

$$G_\varepsilon(v) = (I + \varepsilon\Delta^2)^{-1}(\chi|v|^2v + bv\{\alpha[\mathcal{K}_1(\alpha(|v|^2)_x, (|v|^2)_y)]_x + [\mathcal{K}_2(\alpha(|v|^2)_x, (|v|^2)_y)]_y\}).$$

Thus, we consider the following integral equation:

$$u^\varepsilon(t) = \Sigma_\varepsilon(t)u_0^\varepsilon - \mathrm{i}\int_0^t \Sigma_\varepsilon(t-s)G_\varepsilon(u^\varepsilon(s))\,\mathrm{d}s. \tag{16}$$

This is equivalent to the existence of a solution to the regularized problem (13). For $u_0^\varepsilon \in H^2(\mathbb{R}^2)$ and $v \in \mathcal{C}([0, T]; H^2(\mathbb{R}^2))$, we introduce a mapping $\mathcal{T}$ as follows:

$$\mathcal{T}v(t) = \Sigma_\varepsilon(t)u_0^\varepsilon - \mathrm{i}\int_0^t \Sigma_\varepsilon(t-s)G_\varepsilon(v(s))\,\mathrm{d}s.$$

Hence, the integral equation (16) is the fixed-point problem for $\mathcal{T}$.

By the result of Segal in [17], if the mapping G_ε is locally Lipschitzian on $H^2(\mathbb{R}^2)$ then the integral equation (16) with $u_0^\varepsilon \in H^2(\mathbb{R}^2)$ has a unique maximal solution $u^\varepsilon \in \mathcal{C}([0, T^\varepsilon]; H^2(\mathbb{R}^2))$ which satisfies either $T^\varepsilon = +\infty$ or $0 < T^\varepsilon < \infty$ and $\limsup_{t\to T^\varepsilon}\|u^\varepsilon(t)\|_{H^2(\mathbb{R}^2)}^2 = \infty$. On the other hand, mass conservation (14) is valid because of $u^\varepsilon \in \mathcal{C}([0, T^\varepsilon]; H^2(\mathbb{R}^2))$. Thus, we can see that the blow-up in $H^2(\mathbb{R}^2)$ cannot occur. As a result, we have a global unique solution $u^\varepsilon \in \mathcal{C}(\mathbb{R}_+; H^2(\mathbb{R}^2))$ for $u_0^\varepsilon \in H^2(\mathbb{R}^2)$.

Lemma 1. *The mapping* G_ε *is locally Lipschitzian on* $H^2(\mathbb{R}^2)$.

Proof. Let v_1 and v_2 be in $H^2(\mathbb{R}^2)$ with $\|v_1\|_{H^2(\mathbb{R}^2)} + \|v_2\|_{H^2(\mathbb{R}^2)} \leqslant R$. Since it is known that the mapping $v \to |v|^2v$ is locally Lipschitzian on $H^2(\mathbb{R}^2)$, we are interested in only non-local terms. So we define $\phi_{ij} = \mathcal{K}_i\left(\alpha(|v_j|^2)_x, \left(|v_j^2|\right)_y\right)(i, j = 1, 2)$. From proposition 3, we have

$\|\phi_{i1}\|_{L^\infty(\mathbb{R}^2)} \leqslant C_i R$ and from (A.1), we get $\|\phi_{i1} - \phi_{i2}\|_{L^\infty(\mathbb{R}^2)} \leqslant C_i R \|v_1 - v_2\|_{H^1(\mathbb{R}^2)}$. Also, since $(I + \varepsilon\Delta^2)^{-1/2}$ is an isomorphism from $L^2(\mathbb{R}^2)$ to $H^2(\mathbb{R}^2)$, there exists $C_\varepsilon > 0$ such that

$$\|(I + \varepsilon\Delta^2)^{-1}[\alpha(v_1(\phi_{11})_x - v_2(\phi_{12})_x) + (v_1(\phi_{21})_y - v_2(\phi_{22})_y)]\|_{H^2(\mathbb{R}^2)}$$
$$\leqslant C_\varepsilon(|\alpha|C_1 + C_2)R^2\|v_1 - v_2\|_{H^1(\mathbb{R}^2)}. \qquad \square$$

We now relate the regularized Cauchy problem to the original problem by taking

$$u_0^\varepsilon = (I - \varepsilon^s\Delta)^{-1}u_0, \tag{17}$$

where s is a real number and $0 < s < 1/2$. The following estimates show that the $u_0^\varepsilon \in H^2(\mathbb{R}^2)$ and $H^1(\mathbb{R}^2)$ norms of u_0^ε are bounded, independent of $\varepsilon > 0$:

$$\|u_0^\varepsilon\|_{L^2(\mathbb{R}^2)} \leqslant \|u_0\|_{L^2(\mathbb{R}^2)}, \qquad \|\nabla u_0^\varepsilon\|_{L^2(\mathbb{R}^2)} \leqslant \|\nabla u_0\|_{L^2(\mathbb{R}^2)},$$
$$\varepsilon^s\|\Delta u_0^\varepsilon\|_{L^2(\mathbb{R}^2)} \leqslant \|u_0\|_{L^2(\mathbb{R}^2)}. \tag{18}$$

Similar estimates hold for the solutions of the regularized problem at least for small mass initial data.

Proposition 5. *For $u_0 \in H^1(\mathbb{R}^2)$ satisfying*

$$\left(\max\left\{-\frac{\chi}{2}, 0\right\} + 2|b|\left(b_1\alpha^2 + \frac{b_1}{r_1 r_2} + b_2|\alpha|\right)\right)\|u_0\|^2_{L^2(\mathbb{R}^2)} < 1, \tag{19}$$

there exists a constant $\varepsilon_0 = \varepsilon_0\big(\alpha, \chi, b, b_1, b_2, r_1, r_2, \|u_0\|^2_{L^2(\mathbb{R}^2)}\big)$ and a constant $C_0 = C_0\big(\alpha, \chi, b, b_1, b_2, r_1, r_2, \|u_0\|^2_{L^2(\mathbb{R}^2)}\big)$ such that

$$\|u^\varepsilon(t)\|_{H^1(\mathbb{R}^2)} \leqslant C_0, \qquad \forall t \geqslant 0, \qquad \forall \varepsilon \in (0, \varepsilon_0). \tag{20}$$

Proof. We choose $u_0 \in H^1(\mathbb{R}^2)$ which satisfies (19) and set $\mu = \max\{-\chi/2, 0\} + 2|b|(b_1 + b_1 r_1 r_2\alpha^2 + b_2 r_1 r_2|\alpha|)/(r_1 r_2)$ and $k = \mu\|u_0\|^2_{L^2(\mathbb{R}^2)} < 1$. First, using (14), we can prove that $u^\varepsilon(t)$ is uniformly bounded in $L^2(\mathbb{R}^2)$. By (14) and (18) with $\varepsilon_0 = [(1-k)/(2k)]^{1/(1-2s)}$, we obtain

$$\mu\|u^\varepsilon\|^2_{L^2(\mathbb{R}^2)} \leqslant \mu(1 + \varepsilon^{1-2s})\|u_0\|^2_{L^2(\mathbb{R}^2)} \leqslant \frac{1+k}{2} < 1,$$
$$\forall t \geqslant 0 \qquad \text{and} \qquad \forall \varepsilon \in (0, \varepsilon_0). \tag{21}$$

Second, by using (15), we can show that $\nabla u^\varepsilon(t)$ is uniformly bounded in $L^2(\mathbb{R}^2)$. Thus, we apply the proposition 3 inequality and the Ladyzenskaya inequality $\|u^\varepsilon\|^4_{L^4(\mathbb{R}^2)} \leqslant \|u^\varepsilon\|^2_{L^2(\mathbb{R}^2)}\|\nabla u^\varepsilon\|^2_{L^2(\mathbb{R}^2)}$ in (15) [18]:

$$\|\nabla u^\varepsilon\|^2_{L^2(\mathbb{R}^2)} \leqslant \mu\|u^\varepsilon\|^2_{L^2(\mathbb{R}^2)}\|\nabla u^\varepsilon\|^2_{L^2(\mathbb{R}^2)} + E_\varepsilon(0).$$

Then by using equation (21), we get

$$\|\nabla u^\varepsilon\|^2_{L^2(\mathbb{R}^2)} \leqslant 2(1-k)^{-1}E_\varepsilon(0). \tag{22}$$

To find the upper bound of $E_\varepsilon(0)$, we use the inequalities, (18), proposition 3 and the Ladyzenskaya inequality in (15):

$$E_\varepsilon(0) \leqslant C_3\|\nabla u_0\|^2_{L^2(\mathbb{R}^2)}, \tag{23}$$

where $C_3 = \{1 + [\max\{\chi/2, 0\} + 2|b|(b_1\alpha^2 + b_1/(r_1 r_2) + b_2\alpha)]\|u_0\|_{L^2(\mathbb{R}^2)}\}$. Combining equations (22) and (23), we obtain

$$\|\nabla u^\varepsilon(t)\|^2_{L^2(\mathbb{R}^2)} \leqslant \frac{2C_3}{1-k}\|\nabla u_0\|^2_{L^2(\mathbb{R}^2)}, \qquad \forall t \geqslant 0 \qquad \text{and} \qquad \forall \varepsilon \in (0, \varepsilon_0). \tag{24}$$

Hence, u^ε is uniformly bounded in $\mathcal{C}(\mathbb{R}_+; H^1(\mathbb{R}^2))$ because of (21) and (24). $\square$

4. Passing to the limit as $\varepsilon \to 0$

In this section, we will show that a subsequence of $\{u^\varepsilon\}$ converges to v which satisfies the non-local NLS system. Thus, we will pass to the limit as $\varepsilon \to 0$ in (13). First, we assume that u_0 in $H^1(\mathbb{R}^2)$ satisfies (19), so u^ε is uniformly bounded in $H^1(\mathbb{R}^2)$ for $\varepsilon \in (0, \varepsilon_0)$ and satisfies (20). Then, to see that u_t^ε is bounded independently of ε in $H^{-1}(\mathbb{R}^2)$, we use equation (13):

$$\left\|u_t^\varepsilon\right\|_{H^{-1}(\mathbb{R}^2)} \leqslant 2\|u^\varepsilon\|_{H^1(\mathbb{R}^2)} + C_4|\chi|\|u^\varepsilon\|^3_{H^1(\mathbb{R}^2)} + |b|\left\|u^\varepsilon\left\{\alpha\left(\phi_1^\varepsilon\right)_x + \left(\phi_2^\varepsilon\right)_y\right\}\right\|_{H^{-1}(\mathbb{R}^2)}. \tag{25}$$

Now, we show that $u^\varepsilon(\alpha(\phi_1^\varepsilon)_x + (\phi_2^\varepsilon)_y)$ is also bounded independently of ε in $H^{-1}(\mathbb{R}^2)$:

$$\begin{aligned}\left\|u^\varepsilon\left\{\alpha\left(\phi_1^\varepsilon\right)_x + \left(\phi_2^\varepsilon\right)_y\right\}\right\|_{H^{-1}(\mathbb{R}^2)} &= \sup_{\|\varphi\|_{H^1}\leqslant 1}\left|\int_{\mathbb{R}^2} u^\varepsilon\left\{\alpha\left(\phi_1^\varepsilon\right)_x + \left(\phi_2^\varepsilon\right)_y\right\}\varphi\,\mathrm{d}x\,\mathrm{d}y\right| \\ &\leqslant \left[\|\nabla u^\varepsilon\|_{L^2(\mathbb{R}^2)} + \|u^\varepsilon\|_{L^2(\mathbb{R}^2)}\right]\left[|\alpha|\left\|\phi_1^\varepsilon\right\|_{L^\infty(\mathbb{R}^2)} + \left\|\phi_2^\varepsilon\right\|_{L^\infty(\mathbb{R}^2)}\right].\end{aligned}$$

Then using proposition 3, we have

$$\left\|u^\varepsilon\left[\alpha\left(\phi_1^\varepsilon\right)_x + \left(\phi_2^\varepsilon\right)_y\right]\right\|_{H^{-1}(\mathbb{R}^2)} \leqslant 2(|\alpha|C_1 + C_2)\|u^\varepsilon\|^3_{H^1(\mathbb{R}^2)}. \tag{26}$$

As a result of equations (20), (25) and (26), for every $t \geqslant 0$ and $\varepsilon \in (0, \varepsilon_0)$,

$$\|u^\varepsilon(t)\|_{H^1(\mathbb{R}^2)} + \|u_t^\varepsilon(t)\|_{H^{-1}(\mathbb{R}^2)} \leqslant C_0'. \tag{27}$$

Because of this bound, we can extract from $\{u^\varepsilon\}$ a subfamily which converges to $v \in L^\infty(\mathbb{R}_+; H^1(\mathbb{R}^2))$ in the weak star topology and also from $\left\{u_t^\varepsilon\right\}$ a subfamily which converges to $w \in L^\infty(\mathbb{R}_+; H^{-1}(\mathbb{R}^2))$ in the weak star topology, according to the Alaoglu theorem. One can easily see that $w = v_t$. Hence, $v \in \mathcal{C}\left(\mathbb{R}_+; H_w^1(\mathbb{R}^2)\right)$. We still denote this subsequence by $\{u^\varepsilon\}$.

It remains to show that this v is in fact a solution to our Cauchy problem, i.e. it satisfies the initial value $v(0) = u_0$ as well as equation (3). From the way we have regularized the initial data of (3) in (17), $v(0) = u_0$ follows directly. Passing to the limit in (13) is more involved. Since we seek distributional solutions of (3), it suffices to consider the solutions on $[0, T] \times V$ where $V = [-R, R] \times [-R, R]$ with $R, T > 0$ fixed. This follows from a Cantor diagonalization argument applied to all $R, T \in \mathbb{N}$. Then we will have a master subsequence $\{u^\varepsilon\}$ that converges to v for all the choices of R and T in $\mathbb{N}$. Although we fix R and T, depending on the test function, throughout the argument below they are allowed to be arbitrarily large. The sequence $\{u^\varepsilon\}$ converges to v in the weak star topologies and linear operators respect this convergence. As for the nonlinear terms, they are of two types. The local one, $|u^\varepsilon|^2u^\varepsilon$, can be handled via the Aubin–Lions compactness theorem [19] at least in $[0, T] \times V$ since $H^1(V) \hookrightarrow L^p(V)$ compactly for all $p \in [2, \infty)$. Thus, we have

$$u^\varepsilon \to v \quad \text{in} \quad L^p((0, T);\ L^p(V)), \qquad \forall p \in [2, \infty).$$

The non-local terms that involve ϕ_1^ε and ϕ_2^ε require finer analysis. The non-local nonlinearity involves products of forms $u^\varepsilon\left(\phi_1^\varepsilon\right)_x$ and $u^\varepsilon\left(\phi_2^\varepsilon\right)_y$; we can only hope for weak convergence for $\left(\phi_1^\varepsilon\right)_x$ and $\left(\phi_1^\varepsilon\right)_y$ terms. So we need to establish strong convergence for the term u^ε with respect to a good topology. The following three propositions establish these.

Proposition 6. *There exists $\Psi_1, \Psi_2 \in L^\infty(\mathbb{R}_+ \times \mathbb{R}^2)$ such that $\alpha\left(\phi_1^\varepsilon\right)_x + \left(\phi_2^\varepsilon\right)_y \rightharpoonup \alpha(\Psi_1)_x + (\Psi_2)_y$ in the weak star topology of $L^\infty(\mathbb{R}_+; L^q(V))$ for any $q \in [1, 2)$.*

Proof. By proposition 3 and (27), ϕ_1^ε and ϕ_2^ε are uniformly bounded in $L^\infty(\mathbb{R}_+ \times \mathbb{R}^2)$. Thus we can extract from ϕ_1^ε and ϕ_2^ε subsequences which converge to $\Psi_1 \in L^\infty(\mathbb{R}_+ \times \mathbb{R}^2)$ and $\Psi_2 \in L^\infty(\mathbb{R}_+ \times \mathbb{R}^2)$, respectively, in the weak star topology. Now, since $u^\varepsilon \in L^p((0, T); L^p(V))$, we can obtain the fact that the sequences $\left(\phi_1^\varepsilon\right)_x$ and $\left(\phi_2^\varepsilon\right)_y$ are bounded in $L^\infty(\mathbb{R}_+; L^q(V))$ for every $q \in [1, 2)$ and $R > 0$.

We find $\phi_{1,x}$ and $\phi_{2,y}$ in the characteristics ξ_1, ξ_2, η_1 and η_2 by using equation (15) given in [1] as follows:

$$\begin{aligned}\left(\phi_1^\varepsilon\right)_x &= \alpha_1|u^\varepsilon(\xi_1,\xi_2)|^2 + \alpha_2|u^\varepsilon(\eta_1,\eta_2)|^2 + \alpha_3(J_1+J_2) + \alpha_4(J_3+J_4),\\ \left(\phi_2^\varepsilon\right)_y &= \alpha_5|u^\varepsilon(\xi_1,\xi_2)|^2 + \alpha_6|u^\varepsilon(\eta_1,\eta_2)|^2 + \alpha_7(J_1+J_2) + \alpha_8(J_3+J_4),\end{aligned} \tag{28}$$

where

$$J_1(\xi_1,\xi_2) = \int_{\xi_1}^\infty (|u^\varepsilon(\xi_1',\xi_2)|^2)_{\xi_2}\,\mathrm{d}\xi_1', \qquad J_2(\xi_1,\xi_2) = \int_{\xi_2}^\infty (|u^\varepsilon(\xi_1,\xi_2')|^2)_{\xi_1}\,\mathrm{d}\xi_2',$$

$$J_3(\eta_1,\eta_2) = \int_{\eta_1}^\infty (|u^\varepsilon(\eta_1',\eta_2)|^2)_{\eta_2}\,\mathrm{d}\eta_1', \qquad J_4(\eta_1,\eta_2) = \int_{\eta_2}^\infty (|u^\varepsilon(\eta_1,\eta_2')|^2)_{\eta_1}\,\mathrm{d}\eta_2'.$$

The coefficients that appear in (28) are also defined in (A.5). Then we get

$$\begin{aligned}\left\|\left(\phi_1^\varepsilon\right)_x\right\|_{L^q(V)} \leqslant{}& (|\alpha_1|+|\alpha_2|)\|u^\varepsilon\|_{L^{2q}(V)}^2 + |\alpha_3|(\|J_1\|_{L^q(V)} + \|J_2\|_{L^q(V)})\\ &+ |\alpha_4|(\|J_3\|_{L^q(V)} + \|J_4\|_{L^q(V)}).\end{aligned}$$

Because of $1 \leqslant q < 2$ and inequality (20), $\|u^\varepsilon\|_{L^{2q}(V)}$ is uniformly bounded. Thus, we will show that J_i $(i = 1, \ldots, 4)$ are bounded in $L^q(V)$. If $V = [-R_1, R_1]^2$ with respect to coordinates (ξ_1, ξ_2), then its image under the linear transformation $(\xi_1, \xi_2) \to (\eta_1, \eta_2)$ still lies inside $[-R_2, R_2]^2$ for a suitable $R_2 > 0$. This allows us to make the following estimates:

$$\begin{aligned}\|J_1\|_{L^q(V)}^q &= \frac{1}{2r_1}\int_V \left|\frac{\partial}{\partial\xi_2}\int_{\xi_1}^\infty |u^\varepsilon(\xi_1',\xi_2)|^2\,\mathrm{d}\xi_1'\right|^q \mathrm{d}\xi_1\,\mathrm{d}\xi_2\\ &\leqslant \frac{2^q R_1}{r_1}\int_{-R_1}^{R_1}\left(\int_{-\infty}^\infty |u_{\xi_2}^\varepsilon|^2\,\mathrm{d}\xi_1'\right)^{\frac{q}{2}}\left(\int_{-\infty}^\infty |u^\varepsilon|^2\,\mathrm{d}\xi_1'\right)^{\frac{q}{2}}\mathrm{d}\xi_2\\ &\leqslant \frac{2^q R_1}{r_1}\left(\int_{-R_1}^{R_1}\int_{-\infty}^\infty |u_{\xi_2}^\varepsilon|^2\,\mathrm{d}\xi_1'\,\mathrm{d}\xi_2\right)^{\frac{q}{2}}\left(\int_{-R_1}^{R_1}\left(\int_{-\infty}^\infty |u^\varepsilon|^2\,\mathrm{d}\xi_1'\right)^{\frac{q}{2-q}}\mathrm{d}\xi_2\right)^{\frac{2-q}{2}}.\end{aligned}$$

Since for $u^\varepsilon \in H^1(\mathbb{R}^2)$, we have the classical trace estimate

$$\begin{aligned}\|J_1\|_{L^q(V)}^q &\leqslant 2^{\frac{3q}{2}} R_1 r_1^{\frac{q-2}{2}} \|u^\varepsilon\|_{H^1(\mathbb{R}^2)}^q \left(\int_{-R_1}^{R_1}\left(\int_{-\infty}^\infty\int_{-\infty}^{\xi_2}(|u^\varepsilon|^2)_{\xi_2'}\,\mathrm{d}\xi_2'\,\mathrm{d}\xi_1'\right)^{\frac{q}{2-q}}\mathrm{d}\xi_2\right)^{\frac{2-q}{2}}\\ &\leqslant 2^{\frac{3q+2}{2}} R_1^{\frac{4-q}{2}} r_1^{\frac{q-2}{2}} \|u^\varepsilon\|_{H^1(\mathbb{R}^2)}^q \left[\left(\int_{\mathbb{R}^2}|u^\varepsilon|^2\,\mathrm{d}\xi_2'\,\mathrm{d}\xi_1'\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^2}|(u^\varepsilon)_{\xi_2'}|^2\,\mathrm{d}\xi_2'\,\mathrm{d}\xi_1'\right)^{\frac{1}{2}}\right]^{\frac{q}{2}}\\ &\leqslant 2^{2q+1} R_1^{\frac{4-q}{2}} r_1^{q-1}\|u^\varepsilon\|_{H^1(\mathbb{R}^2)}^{2q},\end{aligned}$$

similarly, $\|J_3\|_{L^q(V)}^q \leqslant 2^{2q+1} R_2^{\frac{4-q}{2}} r_2^{q-1}\|u^\varepsilon\|_{H^1(\mathbb{R}^2)}^{2q}$. The same calculations can be performed for J_2 and J_4 and then $\left(\phi_2^\varepsilon\right)_y$. Hence, $\left(\phi_1^\varepsilon\right)_x$ and $\left(\phi_2^\varepsilon\right)_y$ are bounded in $L^q(V)$. Thus, $\left(\phi_1^\varepsilon\right)_x \rightharpoonup (\Psi_1)_x$ and $\left(\phi_2^\varepsilon\right)_y \rightharpoonup (\Psi_2)_y$ in the weak star topology of $L^\infty(\mathbb{R}_+; L^q(V))$. □

At this point, using all the previous weak convergence results, we can pass to the limit as $\varepsilon \to 0$ in (13):

$$\mathrm{i}v_t + \Delta v = \chi|v|^2 v + bv[\alpha(\Psi_1)_x + (\Psi_2)_y]. \tag{29}$$

We still need to show that $\Psi_1 = \mathcal{K}_1(\alpha(|v|^2)_x, (|v|^2)_y)$ and $\Psi_2 = \mathcal{K}_2((\alpha(|v|^2)_x, (|v|^2)_y)$, in order to conclude that v solves the original equation (3) in the distributional sense. To this end, we start with a strong convergence result for the sequence $\{u^\varepsilon\}$.

Proposition 7. *The sequence* $\{u^\varepsilon\}$ *converges strongly to* v *in* $L^2((0,T); L^2(\mathbb{R}^2))$.

Proof. By using (29), we get $\|v_t\|_{H^{-1}(\mathbb{R}^2)} \leqslant 2\|v\|_{H^1(\mathbb{R}^2)} + C_4|\chi|\|v\|^3_{H^1(\mathbb{R}^2)} + |b|[\|\nabla v\|_{L^2(\mathbb{R}^2)} + \|v\|_{L^2(\mathbb{R}^2)}][|\alpha|\|\Psi_1\|_{L^\infty(\mathbb{R}^2)} + \|\Psi_2\|_{L^\infty(\mathbb{R}^2)}]$. So we obtain $v_t \in L^\infty(\mathbb{R}_+; H^{-1}(\mathbb{R}^2))$, since $v \in \mathcal{C}(\mathbb{R}_+; H^1(\mathbb{R}^2))$ and $\Psi_1, \Psi_2 \in L^\infty(\mathbb{R}_+ \times \mathbb{R}^2)$. Hence, the distributional time derivative of $\int_{\mathbb{R}^2} |v(t,x,y)|^2\,\mathrm{d}x\,\mathrm{d}y$ is equal to $2\mathrm{R}e\langle v, v_t\rangle$, where $\langle , \rangle$ denotes the duality pairing between $H^1(\mathbb{R}^2)$ and $H^{-1}(\mathbb{R}^2)$. Knowing that Ψ_1 and Ψ_2 are real-valued functions and using equation (29), we have $\mathrm{R}e\langle v, v_t\rangle = 0$ which leads to $\|v(t)\|_{L^2(\mathbb{R}^2)} = \|u_0\|_{L^2(\mathbb{R}^2)}$ for all $t \geqslant 0$.

On the other hand, using (14) and (18), we have

$$\int_{\mathbb{R}^2} (|u^\varepsilon|^2 + \varepsilon|\Delta u^\varepsilon|^2)\,\mathrm{d}x\,\mathrm{d}y \leqslant \|u_0\|^2_{L^2(\mathbb{R}^2)} + \varepsilon^{1-2s}\|u_0\|^2_{L^2(\mathbb{R}^2)}. \tag{30}$$

Then we integrate (30) with respect to t for every $T > 0$:

$$\left|\int_0^T \int_{\mathbb{R}^2} (|u^\varepsilon|^2 + \varepsilon|\Delta u^\varepsilon|^2)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t - T\int_{\mathbb{R}^2} |u_0|^2\,\mathrm{d}x\,\mathrm{d}y\right| \leqslant T\varepsilon^{1-2s}\|u_0\|^2_{L^2(\mathbb{R}^2)},$$

which means

$$\lim_{\varepsilon\to 0} \int_0^T \int_{\mathbb{R}^2} (|u^\varepsilon|^2 + \varepsilon|\Delta u^\varepsilon|^2)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t = \int_0^T \int_{\mathbb{R}^2} |v(t)|^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t. \tag{31}$$

Since $(u^\varepsilon, \sqrt{\varepsilon}\Delta u^\varepsilon)$ converges weakly to $(v, 0)$ in $L^2((0,T) \times \mathbb{R}^2)$ and equation (31), $\{u^\varepsilon\}$ converges strongly to v in $L^2((0,T); L^2(\mathbb{R}^2))$. □

Proposition 8. $\Psi_1 = \mathcal{K}_1\left(\alpha(|v|^2)_x, (|v|^2)_y\right)$ *and* $\Psi_2 = \mathcal{K}_2\left((\alpha(|v|^2)_x, (|v|^2)_y\right)$.

Proof. We take $\zeta \in \mathcal{D}((0,\infty) \times \mathbb{R}^2)$ as a real-valued test function with support in $[0,T] \times V$ and use equation (A.1):

$$\begin{aligned}\int_0^T \int_V \phi_1^\varepsilon(t,x,y)\zeta(t,x,y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\ = 2\int_0^T \int_V \int_{\mathbb{R}^2} \mathrm{Re}(\alpha K_1(x',y',x,y)u_x^\varepsilon(t,x',y')u^{\varepsilon *}(t,x',y') \\ + K_2(x',y',x,y)u_y^\varepsilon(t,x',y')u^{\varepsilon *}(t,x',y'))\zeta(t,x,y)\,\mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t.\end{aligned} \tag{32}$$

Because of proposition 7, we can pass to the limit as $\varepsilon \to 0$ in (32):

$$\begin{aligned}\int_0^T \int_V \Psi_1(t,x,y)\zeta(t,x,y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t = \int_0^T \int_V \int_{\mathbb{R}^2} [\alpha K_1(x',y',x,y)(|v(t,x',y')|^2)_x \\ + K_2(x',y',x,y)(|v(t,x',y')|^2)_y]\zeta(t,x,y)\,\mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t.\end{aligned}$$

A similar calculation can be performed for Ψ_2. □

Finally, the global existence of solutions to (1) can be expressed as follows.

Theorem 1. *For* $r_1 \neq r_2$, $1 \leqslant q < 2$, *and for every* $u_0 \in H^1(\mathbb{R}^2)$ *such that*

$$\left(\max\left\{-\frac{\chi}{2}, 0\right\} + 2|b|\left(b_1\alpha^2 + \frac{b_1}{r_1 r_2} + b_2|\alpha|\right)\right)\|u_0\|^2_{L^2(\mathbb{R}^2)} < 1,$$

there exist u, ϕ_1 *and* ϕ_2 *with*

$$u \in L^\infty(\mathbb{R}_+; H^1(\mathbb{R}^2)) \cap \mathcal{C}\big(\mathbb{R}_+; H^1_w(\mathbb{R}^2)\big),$$

$$\phi_1, \phi_2 \in L^\infty(\mathbb{R}_+; \mathcal{C}_b(\mathbb{R}^2)), \qquad \nabla\phi_1, \nabla\phi_2 \in L^\infty\big(\mathbb{R}_+; L^q_{loc}(\mathbb{R}^2)\big)$$

that satisfy (1) in the sense of distributions.

Remark. Theorem 1 is valid for $r_1 = r_2$, the decoupled system, if the small initial condition is replaced by $[\max\{-\chi/2, 0\} + 2|b|(\alpha^2+1)]\|u_0\|^2_{L^2(\mathbb{R}^2)} < 1$ (see [5]).

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Appendix

The integral representation of the solution to the coupled wave equations $(1)_2$ and $(1)_3$, when $r_1 \neq r_2$, is obtained in [1]:

$$\phi_1(x, y) = \mathcal{K}_1(\alpha(|u|)_x, (|u|)_y), \qquad \phi_2(x, y) = \mathcal{K}_2(\alpha(|u|)_x, (|u|)_y), \tag{A.1}$$

where the integral operators $\mathcal{K}_1$ and $\mathcal{K}_2$ are defined as

$$\mathcal{K}_1(f, g)(x, y) = \int_{\mathbb{R}^2} [K_1(x', y', x, y) f(x', y') + K_2(x', y', x, y) g(x', y')]\, \mathrm{d}x'\, \mathrm{d}y',$$

$$\mathcal{K}_2(f, g)(x, y) = \int_{\mathbb{R}^2} [K_2(x', y', x, y) f(x', y') + K_3(x', y', x, y) g(x', y')]\, \mathrm{d}x'\, \mathrm{d}y',$$

with the kernel functions $K_i(x', y', x, y)$ $(i = 1, 2, 3)$ given by

$$\begin{aligned}
K_1(x', y', x, y) = {} & [-c_1c_2 H(-r_1(x'-x) + y' - y) H(r_1(x'-x) + y' - y) \\
& - a_1a_2 H(r_4(r_2(x'-x) + y' - y)) H(-r_6(r_1(x'-x) + y' - y)) \\
& - a_1a_2 H(r_3(-r_2(x'-x) + y' - y)) H(r_5(r_1(x'-x) + y' - y))]\big/\big(2c_g^2\big), \\
K_2(x', y', x, y) = {} & c_2a_2[H(-r_1(x'-x) + y' - y) H(r_1(x'-x) + y' - y) \\
& + H(r_4(r_2(x'-x) + y' - y)) H(-r_6(r_1(x'-x) + y' - y)) \\
& - H(r_3(-r_2(x'-x) + y' - y)) H(r_5(r_1(x'-x) + y' - y))]\big/\big(2c_g^2\big), \\
K_3(x', y', x, y) = {} & [-a_1a_2 H(-r_1(x'-x) + y' - y) H(r_1(x'-x) + y' - y) \\
& - c_1c_2 H(r_4(r_2(x'-x) + y' - y)) H(-r_6(r_1(x'-x) + y' - y)) \\
& - c_1c_2 H(r_3(-r_2(x'-x) + y' - y)) H(r_5(r_1(x'-x) + y' - y))]\big/\big(2c_g^2 r_1 r_2\big),
\end{aligned}$$

where $a_1 = \big(c_g^2 - c_1^2\big)^{1/2}$, $a_2 = \big(c_g^2 - c_2^2\big)^{1/2}$, $r_3 = 2r_1/(r_1 + r_2)$, $r_4 = 2r_1/(r_1 - r_2)$, $r_5 = 2r_2/(r_1 + r_2)$ and $r_6 = 2r_2/(r_1 - r_2)$.

The integral forms of the solution, ϕ_1 and ϕ_2, are written in terms of (ξ_1, ξ_2) as

$$\phi_1(\xi_1, \xi_2) = [-c_1c_2I_1 + a_1a_2(I_2 + I_3)]/(2a_2c_1), \qquad \phi_2(\xi_1, \xi_2) = (I_1 - I_2 + I_3)/(2r_1), \tag{A.2}$$

where

$$I_1 = \frac{a_2}{2c_g^2 r_1} \int_{\mathbb{R}^2} H(\xi_1' - \xi_1) H(\xi_2' - \xi_2)[c_1 f(\xi_1', \xi_2') - a_2 g(\xi_1', \xi_2')]\, \mathrm{d}\xi_2'\, \mathrm{d}\xi_1',$$

$$I_2 = \int_{\mathbb{R}^2} H(\xi_1' - \xi_1 + r_8(\xi_2' - \xi_2)) H(-r_6(\xi_2' - \xi_2)) h_1(\xi_1', \xi_2')\, \mathrm{d}\xi_1'\, \mathrm{d}\xi_2',$$

$$I_3 = \int_{\mathbb{R}^2} H(\xi_1' - \xi_1 + r_7(\xi_2' - \xi_2)) H(r_5(\xi_2' - \xi_2)) h_2(\xi_1', \xi_2')\, \mathrm{d}\xi_1'\, \mathrm{d}\xi_2',$$

with $r_7 = (r_1 - r_2)/(r_1 + r_2)$, $r_8 = (r_1 + r_2)/(r_1 - r_2)$ and

$$h_1 = \frac{c_2}{2c_g^2 r_2}(-a_1 f + c_2 g), \qquad h_2 = -\frac{c_2}{2c_g^2 r_2}(a_1 f + c_2 g). \tag{A.3}$$

The expressions $J_k (k = 1, \ldots, 7)$ are given by

$$J_1(v) = \int_{\mathbb{R}^2} \int_{\xi_2}^{\infty} \int_{\xi_1}^{\infty} |v(\xi_1, \xi_2')||f(\xi_1', \xi_2)|\, \mathrm{d}\xi_1'\, \mathrm{d}\xi_2'\, \mathrm{d}\xi_1\, \mathrm{d}\xi_2,$$

$$J_2(v) = \int_{\mathbb{R}^2} \int_{\xi_2}^{\infty} \int_{\xi_1}^{\infty} |v(\xi_1, \xi_2')||g(\xi_1', \xi_2)|\, \mathrm{d}\xi_1'\, \mathrm{d}\xi_2'\, \mathrm{d}\xi_1\, \mathrm{d}\xi_2,$$

$$J_3(v) = \int_{\mathbb{R}^3} \int_{\xi_1}^{\infty} H(-r_6(\xi_2' - \xi_2))|v(\xi_1', \xi_2)||h_1(\xi_1 - r_8(\xi_2' - \xi_2), \xi_2')|\, \mathrm{d}\xi_1'\, \mathrm{d}\xi_2'\, \mathrm{d}\xi_1\, \mathrm{d}\xi_2,$$

$$J_4(v) = \int_{\mathbb{R}^3} \int_{\xi_2}^{\infty} H(-r_6(\xi_1' - \xi_1))|v(\xi_1, \xi_2')||h_2(\xi_1', \xi_2 - r_8(\xi_1' - \xi_1))|\, \mathrm{d}\xi_1'\, \mathrm{d}\xi_2'\, \mathrm{d}\xi_1\, \mathrm{d}\xi_2,$$

$$J_5(v) = \int_{\mathbb{R}^3} \int_{\xi_2}^{\infty} H(r_5(\xi_1' - \xi_1))|v(\xi_1, \xi_2')||h_1(\xi_1', \xi_2 - r_7(\xi_1' - \xi_1))|\, \mathrm{d}\xi_1'\, \mathrm{d}\xi_2'\, \mathrm{d}\xi_1\, \mathrm{d}\xi_2,$$

$$J_6(v) = \int_{\mathbb{R}^3} \int_{\xi_1}^{\infty} H(r_5(\xi_2' - \xi_2))|v(\xi_1', \xi_2)||h_2(\xi_1 - r_7(\xi_2' - \xi_2), \xi_2')|\, \mathrm{d}\xi_1'\, \mathrm{d}\xi_2'\, \mathrm{d}\xi_1\, \mathrm{d}\xi_2,$$

$$J_7 = \int_{\mathbb{R}^4} H(-r_6(\xi_1' - \xi_1)) H(-r_6(\xi_2' - \xi_2)) \times |h_1(\xi_1 - r_8(\xi_2' - \xi_2), \xi_2')||h_2(\xi_1', \xi_2 - r_8(\xi_1' - \xi_1))|\, \mathrm{d}\xi_1'\, \mathrm{d}\xi_2'\, \mathrm{d}\xi_1\, \mathrm{d}\xi_2. \tag{A.4}$$

The coefficients $\alpha_j (j = 1, \ldots, 8)$ are given by

$$\alpha_1 = \frac{c_2}{2c_g^2}(\alpha c_1 r_1 - a_2), \qquad \alpha_2 = \frac{a_2}{2c_g^2}(\alpha a_1 r_2 + c_2), \qquad \alpha_3 = \frac{c_2}{4c_g^2}(\alpha c_1 r_1 + a_2),$$

$$\alpha_4 = \frac{a_2}{4c_g^2}(\alpha a_1 r_2 - c_2), \qquad \alpha_5 = \frac{a_2}{2c_g^2 r_1^2}(\alpha c_1 r_1 - a_2), \qquad \alpha_6 = -\frac{c_2}{2c_g^2 r_2^2}(\alpha a_1 r_2 + c_2),$$

$$\alpha_7 = \frac{a_2}{4c_g^2 r_1^2}(\alpha c_1 r_1 + a_2), \qquad \alpha_8 = -\frac{c_2}{4c_g^2 r_2^2}(\alpha a_1 r_2 - c_2). \tag{A.5}$$

References

[1] Eden A, Erbay S and Hacinliyan I 2009 Reducing a generalized Davey–Stewartson system to a non-local nonlinear Schrödinger equation *Chaos Solitons Fractals* doi:10.1016/j.chaos.2007.11.035 (to be published)

[2] Babaoglu C and Erbay S 2004 Two-dimensional wave packets in an elastic solid with couple stresses *Int. J. Non-Linear Mech.* **39** 941–9

[3] Babaoglu C, Eden A and Erbay S 2004 Global existence and nonexistence results for a generalized Davey–Stewartson system *J. Phys. A: Math. Gen.* **37** 11531–46

[4] Erofeyev V I and Potapov A I 1993 Longitudinal stain waves in non-linearly-elastic media with couple stresses *Int. J. Non-Linear Mech.* **28** 483–8

[5] Ghidaglia J M and Saut J C 1990 On the initial value problem for the Davey–Stewartson systems *Nonlinearity* **3** 475–506

[6] Linares F and Ponce G 1993 On the Davey–Stewartson systems *Ann. Inst. Henri Poincare Anal. Nonlinéaire* **10** 523–48

[7] Tsutsumi M 1994 Decay of weak solutions to the Davey–Stewartson systems *J. Math. Anal. Appl.* **182** 680–704

[8] Hayashi N and Saut J C 1995 Global existence of small solutions on the Davey–Stewartson system and Ishimori systems *Differ. Integral Eqns* **8** 1657–75

[9] Hayashi N and Hirata H 1996 Global existence and asymptotic behaviour in time of small solutions to the elliptic-hyperbolic Davey–Stewartson system *Nonlinearity* **9** 1387–409

[10] Hayashi N 1996 Local existence in time of small solutions to the Davey–Stewartson systems *Ann. Inst. Henri Poincare* A **65** 313–66

[11] Hayashi N and Hirata H 1997 Local existence in time of small solutions to the elliptic-hyperbolic Davey–Stewartson system in the usual Sobolev space *Proc. Edinburgh Math. Soc.* **40** 563–81

[12] Hayashi N 1997 Local existence in time of solutions to the elliptic-hyperbolic Davey–Stewartson systems with smallness condition on the data *J. Anal. Math.* **73** 133–64

[13] Sulem C and Sulem P L 1999 *The Nonlinear Schrödinger Equation, Self-Focusing and Wave Collapse* (New York: Springer)

[14] Chihara H 1999 The initial value problem for the elliptic-hyperbolic Davey–Stewartson equation *J. Math. Kyoto Univ.* **39** 41–66

[15] Hayashi N and Naumkin P I 1999 On the Davey–Stewartson and Ishimori systems *Math. Phys. Anal. Geom.* **2** 53–81

[16] Hayashi N, Naumkin P I and Uchida H 2002 Analytic smoothing effects of global small solutions to the elliptic-hyperbolic Davey–Stewartson system *Adv. Differ. Eqns* **7** 469–92

[17] Segal I 1963 Non-linear semi-groups *Ann. Math.* **78** 339–64

[18] Foias C, Manley O, Rosa R and Temam R 2004 *Navier-Stokes Equations and Turbulence* (Cambridge: Cambridge University Press)

[19] Lions J L 1969 *Quelques Mèthodes de Rèsolution des Problèmes aux Limites non Linéaires* (Paris: Dunod)